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THE GENERAL SOLUTION OF THE INDETERMINATE EQUATION: $Ax + By + Cz + \ldots\ldots = r$.

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Little has been done since Jacobi (*Werke*, **6**, 355) in connection with the solution of the general linear indeterminate equation. Jacobi has given no less than four methods all involving the reduction of the equation by means of a set of auxiliary equations to the solution of an equation in two variables the solution of which is immediately obtained from the theory of the ordinary continued fraction. The solution here presented treats the general equation in the same non-tentative way that is found in the continued fraction solution for two variables. The method applies equally well when the right hand member is zero and gives a perfectly general solution from which all other special solutions may be obtained.

Consider the set of positive or negative, non-zero integers a_1, b_1, c_1, . . ., k_1, l_1 and let m be the number of terms in the set. We derive from this first set a second set a_2, b_2, c_2, . . . k_2, l_2 by means of the equations:

$$\begin{aligned} a_2 &= b_1 - \alpha_1 a_1, \\ b_2 &= c_1 - \beta_1 a_1 \\ &\ldots\ldots\ldots \\ k_2 &= l_1 - \kappa_1 a_1, \\ l_2 &= a_1 \end{aligned}$$

where if a_1 is different from unity the numbers α_1, β_1, . . . κ_1 are so taken that a_2, b_2, c_2, . . . k_2 are the smallest positive residues, not zero, of the numbers b_1, c_1, . . . k_1, l_1 respectively with respect to the modulus a_1. If a_1 is unity then α_1, β_1 . . . κ_1 are taken equal to b_1, c_1, . . . l_1 respectively, so that in this case the second set consists of zeros with the exception of the last term, l_2 which is unity.

We derive in the same way a third set from the second by means of the equations:

$$\begin{aligned} a_3 &= b_2 - \alpha_2 a_2 \\ b_3 &= c_2 - \beta_2 a_2 \\ &\ldots\ldots\ldots \\ k_3 &= l_2 - \kappa_2 a_2 \\ l_3 &= a_2 \end{aligned}$$

As in the preceding set, if a_2 is different from unity, we take the numbers α_2, β_2, . . . κ_2 so that a_3, b_3, . . . k_3 are the smallest positive, non-zero residues of b_2, c_2, . . . l_2 respectively, modulo a_2. If a_2 is unity, α_2, β_2, . . . κ_2 are taken equal respectively to b_2, c_2, . . . l_2, and in this case the third set consists of zeros with the exception of the last, l_3, which is unity.

Continuing this process, if the original set had no common factor other than unity, we must arrive at a set in which the first number is unity. For it is clear that $a_{n+1} < a_n$ except when b_n is divisible by a_n in which case $a_{n+1} = a_n$. Further $a_{n+2} < a_n$ except when both b_n and c_n are divisible by a_n, in which case $a_{n+2} = a_{n+1} = a_n$, and so on. If now the original set had the greatest common divisor unity so will also the set $a_n, b_n, c_n, \ldots k_n, l_n$, and so not all the numbers $b_n, c_n, \ldots k_n, l_n$ can be divisible by a_n. After a number of steps in the process at most equal to $n - 1$ an a_λ must appear which is less than a_n and not zero. In the same way another set must appear in which the first number is less than a_λ and so on. This process must then lead to a set in which the first number is unity. By taking one more step a set is then obtained in which the numbers are all zeros except the last which is unity.

We proceed now to reverse the above process, and taking the numbers $\alpha_1, \beta_1, \ldots \kappa_1$: $\alpha_2, \beta_2, \ldots \kappa_2$: . . . as given we will show how to reconstruct the original set $a_1, b_1, c_1, \ldots k_1, l_1$ from them. We construct first a determinant of order m in which the element of the prinicipal diagonal are all units, and all the other elements are zero. Using the first set of numbers $\alpha_1, \beta_1, \ldots \kappa_1$, which we will call the first 'partial quotient set' we construct what we will call the 'first determinant' as follows: The top row of the above determinant is erased and another row is added at the bottom which has for its elements $1, \alpha_1, \beta_1, \ldots \kappa_1$. This bottom row we will call the first 'convergent set' and for uniformity of notation we write it $A_1, B_1, C_1, \ldots K_1, L_1$. It is seen that the value of the first determinant is unity.

Using the second partial quotient set, $\alpha_2, \beta_2, \ldots \kappa_2$. we obtain from the first determinant a second determinant by erasing the top row and adding for the bottom row the second convergent set $A_2, B_2, C_2, \ldots K_2, L_2$, the elements of which are obtained by adding the columns of the first determinant after multiplying the first row by 1, the second by α_2, the third by β_2, etc. and the bottom row by κ_2. In the same way we get the third determinant and the third convergent set, using the third partial quotient set $\alpha_3, \beta_3, \ldots \kappa_3$. The nth convergent set, which is the bottom row of the nth determinant is related to the preceding sets by the recursion formulae:

$$A_n = \kappa_n A_{n-1} + \ldots \alpha_n A_{n-m+1} + A_{n-m}$$

with similar formulae for the B_n, C_n, etc. It is clear from the way the determinants are derived from each other and from the original determinant that the value of each is unity.

We state now the remarkable theorem that the last convergent set is identical with the original set $a_1, b_1, c_1 \ldots k_1, l_1$, from which the successive partial quotient sets were derived. This theorem comes out of the general theory of continued fractions, of multiplicity m, but without any appeal to that theory Professor Frank Irwin has derived it very simply from the following equations which are easily established by complete induction:

$$
\begin{array}{ll}
a_1 = a_nA_{n-m} + b_nA_{n-m+1} + \ldots k_nA_{n-1} + l_nA_n & n = 1, 2, \ldots \\
b_1 = a_nB_{n-m} + b_nB_{n-m+1} + \ldots k_nB_{n-1} + l_nB_n & A_0 = A_{-1} = A_{-2} = \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots = A_{-m+1} = 0, \\
l_1 = a_nL_{n-m} + b_nL_{n-m+1} + \ldots k_nL_{n-1} + l_nL_n & A_{-m} = 1, \text{etc.}
\end{array}
$$

Now, as we have seen, in the last set a_n, b_n, c_n, k_n are all zero, and l_n is equal to unity. This gives the theorem.

This theorem throws into our hands a straightforward method of writing down a determinant equal to unity whose bottom row is any given set of positive or negative non-zero integers, with greatest common divisor unity. Let this last row be A_n, B_n, C_n, . . . K_n, L_n, and let the co-factors of these elements in the last determinant be A'_n, B'_n, C'_n, . . . K'_n, L'_n. Then these co-factors furnish a set of values for the unknown in the indeterminate equation $A_nx + B_ny + \ldots L_nv = 1$, and by multiplying these values through by r we get a solution of the equation when the right member is equal to r. Moreover since $A_nA'_\rho + B_nB'_\rho + C_nC'_\rho + \ldots K_nK'_\rho + L_nL'_\rho = 0$ for $\rho = n - 1, n - 2 \ldots n - m + 1$ we may write for the most general value of the variables:

$$
\begin{array}{ll}
x = rA'_n + sA'_{n-1} + tA'_{n-2} + \ldots & +pA'_{n-m+1} + qA'_{n-m} \\
y = rB'_n + sB'_{n-1} + tB'_{n-2} + & pB'_{n-m+1} + qB'_{n-m} \\
z = rC'_n + sC'_{n-1} + tC'_{n-2} + & +pC'_{n-m+1} + qC'_{n-m} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots
\end{array}
$$

That this is the most general form of the solution follows from the fact since the determinant of the co-factors is unity a set of integer values of r, s, t, . . . p, q can always be found for any given set of values of x, y, z, . . .

We give as a numerical example the problem of determining the most general solution of the indeterminate equation:

$$33x + 55y + 79z - 99w = r.$$

The following is a convenient arrangement of the work of computing the sets a_1, b_1, c_1, d_1, α_1, β_1, γ_1, etc.

a_n	b_n	c_n	d_n	α_n	β_n	γ_n
33	55	79	−99	1	2	−4
22	13	33	33	0	1	1
13	11	11	22	0	0	1
11	11	9	13	0	0	1
11	9	2	11	0	0	0
9	2	11	11	0	1	1
2	2	2	9	0	0	4
2	2	1	2	0	0	0
2	1	2	2	0	0	0
1	2	2	2	2	2	2
0	0	0	1			

With these values of α_n, β_n, γ_n we proceed to compute the successive convergent sets and the corresponding determinants. The following is a convenient arrangement of the work:

α_n	β_n	γ_n	A_n	B_n	C_n	D_n
			1	0	0	0
			0	1	0	0
			0	0	1	0
			0	0	0	1
1	2	−4	1	1	2	−4
0	1	1	1	2	2	−3
0	0	1	1	2	3	−3
0	0	1	1	2	3	−2
0	0	0	1	1	2	−4
0	1	1	3	5	7	−9
0	0	4	13	22	31	−39
0	0	0	1	2	3	−2
0	0	0	1	1	2	−4
2	2	2	33	55	79	−99

The last determinant yields us the solution of the proposed problem. Computing the minors we easily get the following general values for the unknowns in the equation:

$$x = -29r - 57s + 54t - 65u$$
$$y = 3r + 4s - 4t + 7u$$
$$z = 5r + 11s - 11t + 11u$$
$$w = -4r - 8s + 7t - 9u$$

. . . .

THE ARCHAEOLOGY OF THE SOUTHWEST: A PRELIMINARY REPORT

By N. C. Nelson

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Introductory.—Since my last communication to the Proceedings of the National Academy (Vol. 3, pp. 192–195) two years ago, concerning the time distribution of aboriginal traits in the Mammoth Cave region of Kentucky, it has become possible to report upon the stratigraphic conditions in two additional North American culture centers. One of these new centers, viz., the Florida-Georgia characterization area, directly adjoins that in which the Mammoth Cave occurs, in fact is in a large sense one with it and need not therefore be specifically discussed. The other center, of special concern here is sepa-